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Estimates for the characteristic problem of the first-order reduction of the wave equation

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Abstract
We calculate certain estimates for the solution of the characteristic problem of the wave equation reduced to first order, in terms of the free data prescribed on two transverse surfaces, one of which is characteristic. Estimates of such kind ensure the stability of the solutions under small variations of the data. Similar estimates exist for the derivatives of the solution as well.

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1. Well posedness and characteristic problems

Given a system of partial differential equations where a unique solution exists for some given data, oftentimes it is necessary to know how sensitive the solution is to small variations of the data. One would not want the equations to amplify any uncertainty in the data beyond control. When, in addition to existence and uniqueness, the equations guarantee the stability of the solution under small perturbations of the data, the problem is referred to as *well posed* [1].

There are two components to a well-posed problem. First there are the equations themselves, which are usually defined only up to transformations of variables and coordinates—respecting the differential order of the equations. Second, there is the data set, usually an appropriate set of values prescribed on a chosen surface. In coordinate-independent terms, the choice of data surface determines the type of problem for the given equations. If one seeks a solution in the interior of a region bounded by a closed data surface, we have a *boundary problem*. If the solution is needed on half of the available space of independent variables ($x^1 \geqslant 0$ in the appropriate choice of coordinates for the problem), then we have an *initial value problem*, or 'Cauchy problem'. It should come as no surprise that some equations have at the same time a well-posed problem and an ill-posed problem. The textbook example is the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$, for which the boundary problem is well posed but the initial value problem is not. A less acknowledged but equally powerful example is the wave equation $u_{xx} - u_{tt} = 0$, for which the reverse statement holds. Typically elliptic equations admit well-posed boundary problems, whereas hyperbolic equations admit well-posed initial value problems.

In the case of hyperbolic equations, there are also *characteristic problems*. In this case, for a unique solution to exist, the data must be prescribed on two intersecting transverse surfaces, one of which is ruled by characteristic curves and we assume the other one is not [2]. On the other hand, characteristic problems are not to be confused with mixed initial-boundary value problems, where values are prescribed on two intersecting surfaces but such that the initial surface is not characteristic, because in such mixed problems the number of data variables necessary to specify a unique solution is larger than in the initial value problem of the same equations. The characteristic problem of a set of equations requires the same number of data as the corresponding initial value problem. In characteristic problems, the method of solution is an adaptation by Duff of the Cauchy–Kowalewsky theorem for initial value problems. As for the stability of the solutions, there does not seem to have been a great deal of interest. Characteristic problems are usually neglected in favour of their very powerful cousins, the initial value problems, which require only one correctly chosen data surface. Still, in some physical contexts such as general relativity, characteristic problems have traditionally been very fruitful in connection with issues of radiation [3], and have been used methodically to obtain numerical solutions [4]. The question of well posedness in such cases becomes quite relevant.

Here the stability of the solutions of the characteristic problem of the wave equation in three spatial dimensions, reduced to first order, is explored by means of a particular method. The method is analogous to the energy estimates used in the case of initial value problems of first-order hyperbolic systems [1, 5]. The principle behind the method is to obtain, as a consequence of the equations, an inequality by which the 'size' of the solution is bounded by the 'size' of the data, namely to estimate the solution in terms of the free data. In linear problems, the estimate applied to the difference between two solutions that are initially close trivially ensures stability.

In section 2 we include a brief expository review of the method for the initial value problem of the wave equation reduced to first order. This is done with the purpose of developing a parallel with either characteristic planes or cones as data surfaces in sections 3 and 4. It is found that a certain kind of estimate for the solution in terms of the data can be established in both cases, demonstrating that these two characteristic problems for the wave equation are well posed. Section 5 contains concluding remarks including a brief overview of the status of the question of stability of characteristic problems in a more general sense. We hope to uphold that even classic problems such as the wave equation have the potential for new insights when viewed from a different angle.

2. The Cauchy problem of the wave equation reduced to first order

We have the wave equation in Cartesian coordinates in three spatial dimensions given by

$$\psi_{tt} = \psi_{xx} + \psi_{yy} + \psi_{zz} \tag{1}$$

where a subscript $x^a \equiv (t, x, y, z)$ denotes partial differentiation with respect to the coordinate x^a, as usual. We do not deal with this equation directly, but prefer to 'reduce' (actually, *extend*) the equation to a system of first-order equations. In doing so, one substitutes the original problem with a problem that has more solutions, among which the solutions to the original problem can be singled out.

We define new variables $U \equiv \psi_t$, $P \equiv \psi_x$, $Q \equiv \psi_y$ and $R \equiv \psi_z$. In terms of these variables, (1) is equivalent to the following system:

$$U_t = P_x + Q_y + R_z, \tag{2a}$$

$$P_t = U_x, \tag{2b}$$

$$Q_t = U_y, \tag{2c}$$

$$R_t = U_z. \tag{2d}$$

$$\psi_t = U, \tag{2e}$$

in the sense that all the solutions $\psi(x^a)$ of (1) are solutions of (2). In order to single them out we need to impose the following constraints on the initial data surface:

$$\mathcal{C}^x \equiv P - \psi_x = 0, \tag{3a}$$

$$\mathcal{C}^y \equiv Q - \psi_y = 0, \tag{3b}$$

$$\mathcal{C}^z \equiv R - \psi_z = 0, \tag{3c}$$

which remain satisfied because their t-derivatives vanish $\left(\mathcal{C}^x{}_t = \mathcal{C}^y{}_t = \mathcal{C}^z{}_t = 0\right)$ by virtue of (2). However, we do not have to restrict the data for the estimates that follow, so we ignore the constraints from now on. In matrix notation the system (2) has the form

$$\boldsymbol{A^a} \partial_a v + \boldsymbol{D} v = 0 \tag{4}$$

where $v = (U, P, Q, R, \psi)$, $\boldsymbol{A^t}$ is the identity matrix, the other three matrices $\boldsymbol{A^i}$ are symmetric, $\boldsymbol{D}$ has all vanishing coefficients but one (of value -1) and summation over repeated indices is understood. Multiplying on the left with v we have

$$v\boldsymbol{A^a} \partial_a v + v\boldsymbol{D} v = 0, \tag{5}$$

and, because the matrices are symmetric, we can extract the partial derivatives to obtain a 'conservation law'

$$\partial_a (v\boldsymbol{A^a} v) + 2v\boldsymbol{D} v = 0. \tag{6}$$

Because we are not interested in a boundary-value problem, we now assume that the fields either decay sufficiently fast at infinity so that their volume integration in the entire space R^3 is finite, or else they are periodic, so that we can restrict attention to a box in R^3. Equivalently, we assume that the fields have a Fourier transform or series. With this assumption we can first split off the time part of the conservation law (6), and then integrate it in the spatial volume $\mathcal{V}$ (denoting either R^3 or a box)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} v\boldsymbol{A^t} v \,\mathrm{d}^3x + \int_{\mathcal{V}} \partial_i (v\boldsymbol{A^i} v) \,\mathrm{d}^3x + 2 \int_{\mathcal{V}} v\boldsymbol{D} v \,\mathrm{d}^3x = 0. \tag{7}$$

With our assumptions, the second term evaluates to zero. In the first term we have the L_2 norm of the fields, because $\boldsymbol{A^t}$ is the identity matrix and thus

$$\int_{\mathcal{V}} v\boldsymbol{A^t} v \,\mathrm{d}^3x = \|v(t, \cdot)\|^2. \tag{8}$$

Since $v\boldsymbol{D}v = -U\psi$, equation (7) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \|v(t, \cdot)\|^2 = \int_{\mathcal{V}} 2U\psi \,\mathrm{d}^3x. \tag{9}$$

Finally, since $2U\psi \leqslant U^2 + \psi^2$ then $\int_{\mathcal{V}} 2U\psi \, \mathrm{d}^3x \leqslant \|v(t,\cdot)\|$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t,\cdot)\|^2 \leqslant \|v(t,\cdot)\|^2 \tag{10}$$

which implies

$$\|v(t,\cdot)\|^2 \leqslant \mathrm{e}^t \|v(0,\cdot)\|^2 \tag{11}$$

and we have our estimate of the 'size' of the solution in terms of the 'size' of the initial data. Since the system is linear, the difference of two solutions is also a solution, so small variations of the initial data result in small variations of the solution, at least for small enough times. The exponential factor is there because of the presence of undifferentiated terms in (2). In spite of the exponential factor, the inequality shows that the solution depends continuously on the data, so that the variation in the solution at any time can be controlled by refining the accuracy of the data. Nonetheless, as usual [5], the inequality is useless for large values of t for numerical purposes, where the interest resides in explicitly estimating the error in a given solution that starts with no less than round-off error.

Alternatively, one can get a better estimate in this case by treating ψ separately from the rest of the variables. Clearly the equations for the variables U, P, Q and R decouple from ψ, and have no undifferentiated terms. A solution can be found independently of the value of ψ, and can be estimated also independently of ψ. If we reproduce the preceding reasoning with $v \equiv (U, P, Q, R)$, then we have equation (4) with $\boldsymbol{D} = \boldsymbol{0}$, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t,\cdot)\|^2 = 0, \tag{12}$$

and consequently

$$\|v(t,\cdot)\|^2 = \|v(0,\cdot)\|^2, \tag{13}$$

without an exponential factor. Then ψ can be estimated by

$$\psi = \psi|_{t=0} + \int_0^t U \, \mathrm{d}t' \tag{14}$$

which is the solution of $\psi_t = U$ for known source U and given initial data at $z = 0$. This leads, for instance, to an estimate of the form

$$\psi^2 \leqslant 2\psi|_{t=0}^2 + 2\left(\int_0^t U \, \mathrm{d}t'\right)^2. \tag{15}$$

Most remarkably, one can also control the smoothness of the solution. The spatial derivatives of the fields have estimates in terms of the spatial derivatives of the initial data. One can see this easily by taking a spatial derivative ∂_i of the system (2*a*)–(2*d*), which thus becomes an evolution system for the variables $v_i = (U_i, P_i, Q_i, R_i)$ of exactly the same kind as that for v, leading to estimates for each spatial derivative v_i in terms of the same spatial derivative of the initial data

$$\|v_i(t,\cdot)\|^2 = \|v_i(0,\cdot)\|^2, \tag{16}$$

for $i = x, y, z$ alternatively.

As a hyperbolic equation, the wave equation admits characteristic lines at 45° with the time axis in all spatial directions from any fixed point, that is all light-like straight lines through any point. A (hyper)surface that has a null (light-like) normal at all points is automatically ruled by characteristic lines and is referred to as a null surface. A null surface is simply the evolution of a wavefront forward in time out of any given initial shape. Any null surface can be chosen as the data surface of a characteristic problem. Thus there are infinite ways to set up a characteristic problem for the wave equation. But there are two aesthetically appealing cases. The first case is that of a null plane acting as the data surface (the evolution of a plane wavefront). Of course, there are null planes in all possible directions, so this may not be useful unless our problem has a particular preferred direction. If there is no particular preferred direction then null cones are suitable (the evolution of a spherical wavefront out of some point of origin). These two choices of characteristic data surfaces lead to similar characteristic problems, which are examined in the following two sections.

3. Well posedness of the characteristic problem of the wave equation with a plane characteristic data surface

The simplest characteristic problem of the wave equation prescribes data on a plane characteristic surface. In addition, we use a plane time-like surface to prescribe the necessary complementary data. In section 3.1 we examine the stability of the solutions under small perturbations of the data. Estimates for the derivatives of the solutions are derived in section 3.2.

3.1. Estimate of the solution

We transform coordinates $(t, x, y, z) \to (u, x, y, z)$ with

$$u = t - z, \tag{17}$$

so that the level surfaces of u are null planes. The wave equation (1) in these coordinates reads

$$2\psi_{uz} - \psi_{xx} - \psi_{yy} - \psi_{zz} = 0. \tag{18}$$

The second derivative with respect to u does not appear in the equation, so we only need to define three additional variables to reduce the equation to a first-order system. We have, as before, $P \equiv \psi_x$, $Q \equiv \psi_y$ and $R \equiv \psi_z$. In terms of these variables, (1) is equivalent to the following system:

$$2R_u = P_x + Q_y + R_z, \tag{19a}$$

$$P_z = R_x, \tag{19b}$$

$$Q_z = R_y, \tag{19c}$$

$$\psi_z = R, \tag{19d}$$

in the sense that all the solutions ψ are contained in the set of solutions of this system. This system has a unique solution in the 'wedge' space with $u \geqslant 0$ and $z \geqslant 0$ if the value of R is prescribed on the null surface $u = 0$ and the values of P, Q and ψ are prescribed on the surface $z = 0$. This is referred to as the canonical form of the characteristic problem of the wave equation [2]. Like all characteristic problems in canonical form, it consists of two sets of equations: one of them internal to each characteristic surface (equations (19*b*)–(19*d*)), and the other moving from one characteristic surface to the next (equation (19*a*)). The field R is a *normal variable*, whereas P, Q and ψ are referred to as *null variables* [2]. We can single out solutions to the original second-order wave equation by choosing data so that the constraints (3*a*) and (3*b*) are satisfied along the surface $z = 0$. They remain satisfied everywhere because $\partial_z \mathcal{C}^x = \partial_z \mathcal{C}^y = 0$ by virtue of (19).

As in the case of the initial value problem in the preceding section, we see that the equations for (R, P, Q) decouple from ψ. Therefore, we can find and estimate a solution independently of ψ. Disregarding equation (19*d*), the rest of the system (19) has the form

$$\boldsymbol{A}^{\boldsymbol{u}} v_{,u} + \boldsymbol{A}^{\boldsymbol{z}} v_{,z} + \boldsymbol{A}^{\boldsymbol{x}} v_{,x} + \boldsymbol{A}^{\boldsymbol{y}} v_{,y} = 0 \tag{20}$$

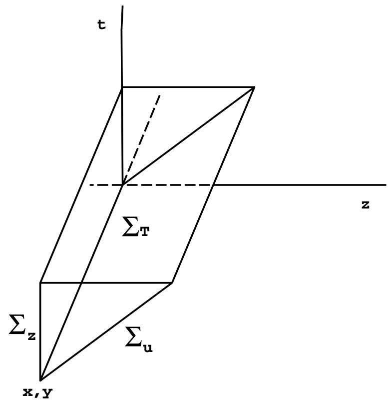

Figure 1. The 'hyper-prism' of integration for the characteristic problem with a null-plane data surface. The top boundary surface Σ_T lies at $u + z = T$ constant and has measure $\mathrm{d}\Sigma_T = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z/\sqrt{2}$. The surface Σ_u is a null plane at $u = 0$ and has measure $\mathrm{d}\Sigma_u = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The surface Σ_z lies at $z = 0$ and has measure $\mathrm{d}\Sigma_z = \mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y$. Both surfaces Σ_u and Σ_z are data surfaces for the characteristic problem.

where $v = (R, P, Q)$, and the matrices are given by

$$\boldsymbol{A}^u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}^z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$\boldsymbol{A}^x = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}^y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{22}$$

Because all the matrices $\boldsymbol{A}^a$ $(a = u, x, y, z)$ are symmetric, it follows that

$$\partial_a(v\boldsymbol{A}^a v) = 0, \tag{23}$$

so we obtain a 'conservation law' of the form $\partial_a J^a = 0$. Integration of this conservation law on any given volume in R^4 (of measure $\mathrm{d}^4x = \mathrm{d}u\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$) gives us relationships between the flows of $v\boldsymbol{A}^a v$ across the surfaces that enclose such volume. We choose our four-volume of integration to be a 'hyper-prism', bounded by the surfaces $u = 0$ (of outward pointing normal $n^1_a = -\delta^u_a$), $z = 0$ (of outward pointing normal $n^2_a = -\delta^z_a$) and $u + z = T$ for a fixed constant T (of outward pointing normal $n^1_a = (\delta^u_a + \delta^z_a)/\sqrt{2}$). Assuming that the fields admit Fourier transforms in the x and y directions, we let the 'hyper-prism' extend to infinity along x and y, or alternatively we can have a periodic boundaries in x and y. Our four-volume of integration $\mathcal{V}^4$ is represented in figure 1. Integrating on this 'hyper-prism' we find

$$\begin{aligned} 0 = \int_{\mathcal{V}^4} \partial_a(v\boldsymbol{A}^a v)\,\mathrm{d}^4x &= \int_{u+z=T} v(\boldsymbol{A}^u + \boldsymbol{A}^z)v\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y - \int_{u=0} v\boldsymbol{A}^u v\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \int_{z=0} v\boldsymbol{A}^z v\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{24}$$

Since, by (21), $\boldsymbol{A}^u + \boldsymbol{A}^z$ is the identity matrix, then $v(\boldsymbol{A}^u + \boldsymbol{A}^z)v = R^2 + P^2 + Q^2$ and its integral on the surface Σ_T may *define* a (positive-definite) norm of the solution,

$$\|v\|^2_T \equiv \int_{u+z=T} v(\boldsymbol{A}^u + \boldsymbol{A}^z)v\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y, \tag{25}$$

which may be used as a measure of the 'size' of the solution generated by data on $u = 0$ and on $z = 0$. On the other hand, $v\boldsymbol{A}^u v = 2R^2$ and $v\boldsymbol{A}^z v = -R^2 + P^2 + Q^2$. With these expressions, (24) can be written as

$$\|v\|_T^2 = 2\int_{\Sigma_u} R^2\,\mathrm{d}\Sigma_u + \int_{\Sigma_z} (-R^2 + P^2 + Q^2)\,\mathrm{d}\Sigma_z \tag{26}$$

which trivially implies the following inequality:

$$\|v\|_T^2 \leqslant 2\left(\int_{\Sigma_u} R^2\,\mathrm{d}\Sigma_u + \int_{\Sigma_z} (P^2 + Q^2)\,\mathrm{d}\Sigma_z\right). \tag{27}$$

The right-hand side is a positive-definite measure of the 'size' of the free data, properly taking into account both data surfaces. This differs markedly from other possible energy-like norms that could be defined, mirroring the Cauchy problem, by integrating the squares of *all fields* on the initial characteristic surface. When all fields are integrated on the surface at a fixed value of u, one is, in a sense, over-counting the null data, thus a literal translation of the energy of the Cauchy problem does not, in fact, play the intended role. In contrast, the inequality (27) functions as the proper characteristic analogue of the standard estimate for initial value problems. Small data R on $u = 0$ and P, Q on $z = 0$ result in small values of R, P and Q within the 'wedge' space $u \geqslant 0$, $z \geqslant 0$.

Once R is known, we can estimate ψ by integrating equation (19d) with given values on $z = 0$

$$\psi = \psi|_{z=0} + \int_0^z R\,\mathrm{d}z', \tag{28}$$

which implies, for instance,

$$\psi^2 \leqslant 2\psi|_{z=0}^2 + 2\left(\int_0^z R\,\mathrm{d}z'\right)^2. \tag{29}$$

3.2. Estimates of the derivatives

It is not as simple to estimate the derivatives of the variables in terms of the derivatives of the data as it is in the case of the Cauchy problem discussed in the previous section. However, with some ingenuity, we can show that the derivatives R_x, R_y, R_z, P_x, P_y, P_u, Q_x, Q_y and Q_u can be controlled with the choice of data. These play a role that is analogous to that of the space derivatives in the case of the Cauchy problem discussed in section 2—P_z, Q_z and R_u are already given by the system of equations (19), and can be estimated directly by the use of the system of equations, as in the case of the Cauchy problem. Of these, R_x, R_y, R_z should be treated as normal variables and the rest as null variables. If so, the data for R_x, R_y, R_z can be obtained from the derivatives of the data for R and we avoid the introduction of more arbitrary data (likewise with the rest of the first-order variables). With this in mind, we obtain equations for R_x, R_y, R_z by applying ∂_i to (19a). Every appearance of P_z or Q_z in the result is substituted in terms of the derivatives of interest by the use of (19b)–(19c). For P_x, Q_x, P_y, Q_y, P_u and Q_u, we apply ∂_x, ∂_y and ∂_u, respectively, to (19b)–(19c) and substitute all appearances of R_u in terms of the other derivatives of interest by the use of (19a). There is some freedom in choosing the ordering of partial derivatives that converts the resulting second-order system into a first-order one. The following results from one of such choices:

$$2\partial_u R_x = \partial_x P_x + \partial_y Q_x + \partial_x R_z, \tag{30a}$$

$$2\partial_u R_y = \partial_x P_y + \partial_y Q_y + \partial_y R_z, \tag{30b}$$

$$2\partial_u R_z = \partial_x R_x + \partial_y R_y + \partial_z R_z, \tag{30c}$$

$$\partial_z P_x = \partial_x R_x, \tag{30d}$$

$$\partial_z Q_x = \partial_y R_x, \tag{30e}$$

$$\partial_z P_y = \partial_x R_y, \tag{30f}$$

$$\partial_z Q_y = \partial_y R_y, \tag{30g}$$

$$2\partial_z P_u = \partial_x P_x + \partial_x Q_y + \partial_x R_z, \tag{30h}$$

$$2\partial_z Q_u = \partial_y P_x + \partial_y Q_y + \partial_y R_z. \tag{30i}$$

This is again a characteristic problem in the canonical form. Note that the first seven equations, (30*a*)–(30*g*), decouple from the last two, since they involve all the variables except P_u or Q_u. Thus R_x, R_y, R_z, P_x, P_y, Q_x and Q_y can be found without knowledge of P_u or Q_u. Once they are known, they can be used as known sources in the right-hand side of (30*h*)–(30*i*) to integrate P_u and Q_u via

$$P_u = P_u|_{z=0} + \frac{1}{2}\partial_x \int_0^z (P_x + Q_y + R_z)\,\mathrm{d}z', \tag{31a}$$

$$Q_u = Q_u|_{z=0} + \frac{1}{2}\partial_y \int_0^z (P_x + Q_y + R_z)\,\mathrm{d}z'. \tag{31b}$$

For this reason, we now restrict attention to the subsystem (30*a*)–(30*g*). It constitutes a characteristic problem of the form

$$\boldsymbol{B}^a \partial_a w = 0, \tag{32}$$

for $w = (R_x, R_y, R_z, P_x, Q_x, P_y, Q_y)$ where the four principal seven-dimensional matrices $\boldsymbol{B}^a$ are symmetric and have constant coefficients, which implies a conservation law

$$\partial_a(w\boldsymbol{B}^a w) = 0. \tag{33}$$

Integrating this conservation law on the 'hyper-prism' we have

$$\int_{u+z=T} w(\boldsymbol{B}^u + \boldsymbol{B}^z)w\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_{u=0} w\boldsymbol{B}^u w\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y + \int_{z=0} w\boldsymbol{B}^z w\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y. \tag{34}$$

Since $w\boldsymbol{B}^u w = 2\left(R_x^2 + R_y^2 + R_z^2\right)$ and $w\boldsymbol{B}^z w = -R_z^2 + P_x^2 + Q_x^2 + P_y^2 + Q_y^2$, this equation reads explicitly

$$\int_{\Sigma_T} \left(2R_x^2 + 2R_y^2 + R_z^2 + P_x^2 + Q_x^2 + P_y^2 + Q_y^2\right)\mathrm{d}\Sigma_T = 2\int_{\Sigma_u} \left(R_x^2 + R_y^2 + R_z^2\right)\mathrm{d}\Sigma_u + \int_{\Sigma_z} \left(-R_z^2 + P_x^2 + Q_x^2 + P_y^2 + Q_y^2\right)\mathrm{d}\Sigma_z, \tag{35}$$

which trivially implies

$$\|w\|_T^2 \leqslant 2\left(\int_{\Sigma_u} \left(R_x^2 + R_y^2 + R_z^2\right)\mathrm{d}\Sigma_u + \int_{\Sigma_z} \left(P_x^2 + Q_x^2 + P_y^2 + Q_y^2\right)\mathrm{d}\Sigma_z\right). \tag{36}$$

Here the norm of the derivatives w is the natural extension of the norm of the solution v to seven dimensions, defined by

$$\|w\|_T^2 \equiv \int_{\Sigma_T} \left(R_x^2 + R_y^2 + R_z^2 + P_x^2 + Q_x^2 + P_y^2 + Q_y^2\right)\mathrm{d}\Sigma_T. \tag{37}$$

The inequality (36) gives us an estimate of the size of the derivatives of interest in terms of the derivatives of the free data, accounting for all the free data without over-counting, in a manner similar to the estimates obtained in the case of the standard initial value problem, except that P_u and Q_u are treated in a special manner, that is as solutions of ordinary differential equations, equations (30h)–(30i). Therefore, the derivatives of the solution of the characteristic problem of the wave equation are *a priori* under control.

4. Well posedness of the characteristic problem of the wave equation with a null cone data surface

The case of data prescribed on a null cone shares all the conceptual features of the case of a null plane. However, it becomes technically much more involved, essentially due to the presence of non-constant coefficients in the expression of the wave equation in spherical coordinates. The estimates of the solution in terms of the data can be derived in a manner similar as the case of a null plane, and are obtained in section 4.1. In order to estimate the derivatives, however, new tools are necessary, which are developed in section 4.2.

4.1. Estimate of the solution

If we wish to prescribe data on a null cone, it is convenient to transform to spherical coordinates $(x, y, z) \to (r, \theta, \phi)$, in which case the wave equation (1) reads

$$\psi_{tt} - \frac{1}{r}\partial_{rr}(r\psi) - \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\,\psi_\theta) - \frac{1}{r^2\sin^2\theta}\psi_{\phi\phi} = 0. \tag{38}$$

For convenience we define the new variable

$$g \equiv r\psi \tag{39}$$

and change θ to a linear coordinate $\theta \to s \equiv \cos\theta$, obtaining

$$g_{tt} - g_{rr} - \frac{1}{r^2}\partial_s((1-s^2)g_s) - \frac{1}{r^2(1-s^2)}g_{\phi\phi} = 0. \tag{40}$$

Now we change coordinates $(t, r, s, \phi) \to (u, r, s, \phi)$ via

$$u = t - r, \tag{41}$$

and the wave equation (40) becomes

$$2g_{ur} - g_{rr} - \frac{1}{r^2}\partial_s((1-s^2)g_s) - \frac{1}{r^2(1-s^2)}g_{\phi\phi} = 0. \tag{42}$$

In order to turn it into a first-order problem we define $R \equiv g_r$, $P \equiv g_s\sqrt{1-s^2}/r$ and $Q \equiv g_\phi/(\sqrt{1-s^2}r)$. We obtain the following system:

$$2R_u = R_r + \frac{\sqrt{1-s^2}}{r}P_s + \frac{1}{r\sqrt{1-s^2}}Q_\phi - \frac{sP}{r\sqrt{1-s^2}} \tag{43a}$$

$$P_r = \frac{\sqrt{1-s^2}}{r}R_s - \frac{P}{r} \tag{43b}$$

$$Q_r = \frac{1}{r\sqrt{1-s^2}}R_\phi - \frac{Q}{r} \tag{43c}$$

$$g_r = R. \tag{43d}$$

For a unique solution to exist in the region of R^4 limited by a worldtube of radius r_0 and a null cone at $u = 0$, this system requires prescribed values of R on the null surface Σ_u at $u = 0$, and values of g, P, Q on the worldtube Σ_r at $r = r_0$. As in the previous case, the variables P, Q, R can be obtained with no knowledge of g, and so can their estimate. Ignoring (43d), the remaining equations in the system (43) have the form

$$\boldsymbol{A}^u v_{,u} + \boldsymbol{A}^r v_{,r} + \boldsymbol{A}^s v_{,s} + \boldsymbol{A}^\phi v_{,\phi} = \boldsymbol{D} v \tag{44}$$

where $v = (R, P, Q)$, and the matrices are given by

$$\boldsymbol{A}^u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}^r = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{45}$$

$$\boldsymbol{A}^s = -\frac{\sqrt{1-s^2}}{r} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{46}$$

$$\boldsymbol{A}^\phi = -\frac{1}{r\sqrt{1-s^2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{47}$$

$$\boldsymbol{D} = -\frac{1}{r} \begin{pmatrix} 0 & s/\sqrt{1-s^2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

There are two difficulties with this characteristic problem that are absent in the case of null-plane data of section 3. First, the principal matrices have non-constant coefficients. This is not an obstacle, though, because, since they are symmetric, we can still obtain a 'conservation law'. The second difficulty is that there are undifferentiated terms, which will appear as sources of the conservation law. Multiplying equation (44) by v on the left, and after combining terms appropriately, we have

$$(v\boldsymbol{A}^u v)_{,u} + (v\boldsymbol{A}^r v)_{,r} + (v\boldsymbol{A}^s v)_{,s} + (v\boldsymbol{A}^\phi v)_{,\phi} = 2v\tilde{\boldsymbol{D}} v, \tag{49}$$

or $\partial_a J^a = S$ with $J^a \equiv v\boldsymbol{A}^a v$. Here we have

$$\tilde{\boldsymbol{D}} = -\frac{1}{r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

Since $v\tilde{\boldsymbol{D}}v = -(P^2 + Q^2)/r$, it follows that

$$\partial_a (v\boldsymbol{A}^a v) \leqslant 0. \tag{51}$$

Thus the presence of the undifferentiated terms in this particular case will not affect the estimate. We can integrate now this law on a spacetime volume $\mathcal{V}^4$, and convert the volume integral of the divergence into a surface integral over the boundary of $\mathcal{V}^4$. We choose a region of integration limited by Σ_u at $u = 0$, Σ_r at $r = r_0$ and Σ_T at a constant value of $u + r = T + r_0$, with $-1 \leqslant s \leqslant 1$ and $0 \leqslant \phi \leqslant 2\pi$. The region of integration is represented in figure 2. We have

$$0 \geqslant \int_{\mathcal{V}^4} \partial_a (v\boldsymbol{A}^a v)\, \mathrm{d}u\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}\phi = \int_{\Sigma_T} v(\boldsymbol{A}^u + \boldsymbol{A}^r)v\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}\phi - \int_{\Sigma_u} v\boldsymbol{A}^u v\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}\phi$$
$$- \int_{\Sigma_r} v\boldsymbol{A}^r v\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}\phi \tag{52}$$

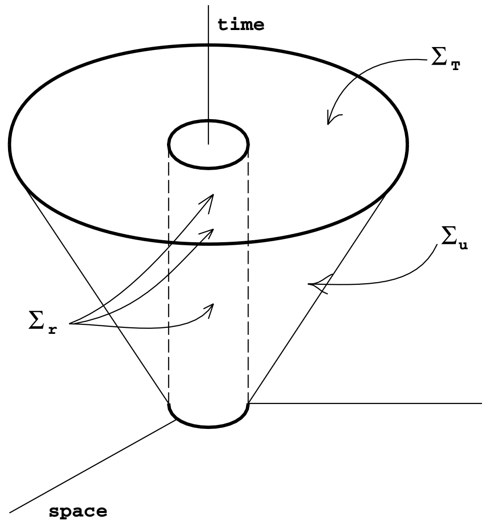

Figure 2. The region of integration for the characteristic problem with a cone data surface. The top boundary surface Σ_T lies at $u + r = T + r_0$ constant and has measure $\mathrm{d}\Sigma_T = \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\phi/\sqrt{2}$. The surface Σ_u is a null cone at $u = 0$ and has measure $\mathrm{d}\Sigma_u = \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\phi$. The surface Σ_r is a worldtube of radius $r = r_0$ and has measure $\mathrm{d}\Sigma_z = \mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\phi$. Both surfaces Σ_u and Σ_r are data surfaces for the characteristic problem.

Similarly as in the case of the null-plane data surface, we define

$$\begin{aligned}\|v\|_T^2 &\equiv \int_{\Sigma_T} (R^2 + P^2 + Q^2)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\phi \\ &= \int_{\Sigma_T} v(\boldsymbol{A}^u + \boldsymbol{A}^r)v\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\phi.\end{aligned} \tag{53}$$

With (53), the inequality (52) reads explicitly

$$\|v\|_T^2 \leqslant 2\int_{\Sigma_u} R^2\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}\phi + \int_{\Sigma_r} (-R^2 + P^2 + Q^2)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\phi \tag{54}$$

which by the same argument as in the case of the null-plane data surface leads to a similar kind of estimate

$$\|v\|_T^2 \leqslant 2\left(\int_{\Sigma_u} R^2\,\mathrm{d}\Sigma_u + \int_{\Sigma_r} (P^2 + Q^2)\,\mathrm{d}\Sigma_r\right). \tag{55}$$

So we see that the 'size' of the data on both data surfaces controls the 'size' of the solution. Finally, once R is known, the variable g can be found and estimated by integrating (43d) with given values on $r = r_0$:

$$g = g|_{r_0} + \int_{r_0}^{r} R\,\mathrm{d}r'. \tag{56}$$

4.2. Estimates of the derivatives

The results of the previous section establish that the size of the solution is controlled by the size of the free data. We wish to be able to control the size of the derivatives

R_r, R_s, R_ϕ, P_u, P_s, P_ϕ, Q_u, Q_s and Q_ϕ of the solution in terms of the derivatives of the free data, as well, since these are the derivatives that play a role analogous to that of the space derivatives in the case of the Cauchy problem. This will be possible if we can write down a characteristic system of equations for these nine variables, where R_r, R_s, R_ϕ will be normal variables and P_u, P_s, P_ϕ, Q_u, Q_s and Q_ϕ will be null variables, as a consequence of the original system of equations. If we write down a system for the derivatives themselves in a manner completely analogous to the method employed in section 3.2, the system (43) implies the following nine equations:

$$2\partial_u R_s - \frac{\sqrt{1-s^2}}{r}\partial_s P_s - \frac{1}{r\sqrt{1-s^2}}\partial_\phi Q_s - \partial_s R_r = -\frac{2sP_s}{r\sqrt{1-s^2}} + \frac{sQ_\phi - (1+s^2)P}{r(1-s^2)^{\frac{3}{2}}}, \tag{57a}$$

$$2\partial_u R_\phi - \frac{\sqrt{1-s^2}}{r}\partial_s P_\phi - \frac{1}{r\sqrt{1-s^2}}\partial_\phi Q_\phi - \partial_\phi R_r = -\frac{sP_\phi}{r\sqrt{1-s^2}}, \tag{57b}$$

$$\begin{aligned}2\partial_u R_r - \frac{1-s^2}{r^2}\partial_s R_s - \frac{1}{r^2(1-s^2)}\partial_\phi R_\phi - \partial_s R_r \\ = \frac{sR_s}{r} - \frac{2\sqrt{1-s^2}P_s}{r^2} + \frac{s(P - rP_r) - 2Q_\phi}{r^2\sqrt{1-s^2}},\end{aligned} \tag{57c}$$

$$\partial_r P_s - \frac{\sqrt{1-s^2}}{r}\partial_s R_s = -\frac{P_s}{r} - \frac{sR_s}{r\sqrt{1-s^2}}, \tag{57d}$$

$$\partial_r Q_s - \frac{1}{r\sqrt{1-s^2}}\partial_\phi R_s = -\frac{Q_s}{r} - \frac{sR_\phi}{r(1-s^2)^{\frac{3}{2}}}, \tag{57e}$$

$$\partial_r P_\phi - \frac{\sqrt{1-s^2}}{r}\partial_s R_\phi = -\frac{P_\phi}{r}, \tag{57f}$$

$$\partial_r Q_\phi - \frac{1}{r\sqrt{1-s^2}}\partial_\phi R_\phi = -\frac{Q_\phi}{r}, \tag{57g}$$

$$2\partial_r P_u - \frac{\sqrt{1-s^2}}{r}\partial_s R_r - \frac{1-s^2}{r^2}\partial_s P_s - \frac{1}{r^2}\partial_s Q_\phi = -\frac{2sP_s}{r^2} + \frac{sQ_\phi - (1+s^2)P}{r^2(1-s^2)} - \frac{2P_u}{r}, \tag{57h}$$

$$2\partial_r Q_u - \frac{1}{r\sqrt{1-s^2}}\partial_\phi R_r - \frac{1}{r^2}\partial_\phi P_s - \frac{1}{r^2(1-s^2)}\partial_\phi Q_\phi = -\frac{sP_\phi}{r^2(1-s^2)} - \frac{2Q_u}{r}. \tag{57i}$$

The first seven equations nither involve P_u nor Q_u; therefore, we may consider equations (57a)–(57g) as a system of seven equations for the seven variables R_r, R_s, R_ϕ, P_s, P_ϕ, Q_s and Q_ϕ, the solution of which can be used as a known source for (57h)–(57i). For the moment, we ignore the last two equations, which we will come back to once we have an estimate from the first seven. The system (57a)–(57g) has many features that make it unsuitable for our purposes. Firstly, the undifferentiated function P is involved, but it is not one of our nine variables. This can be interpreted as a non-homogeneous system of equations for the seven first derivatives of interest, where P acts as a known forcing source. P presence would definitely affect any estimate that may be implied by the system as it stands. Secondly, there are many undifferentiated terms in these equations, making it almost certain that they will play a role in any estimates. Thirdly, the principal matrices are not symmetric, although their asymmetry is not severe because the vanishing coefficients appear symmetrically. As we know, the symmetry of the principal matrices would lead directly to a conservation law with a non-vanishing source of undifferentiated variables, our main goal.

The first obstacle is a true impediment as far as we can see. The standard way to derive estimates for non-homogeneous systems of equations is by means of Duhamel's principle, which allows one to express the solution of the non-homogeneous system in terms of the solution of the associated homogeneous system and forcing source function [5]. We are not aware of any analogous principle for the characteristic problem. Thus, at the moment we are forced to consider only homogeneous characteristic problems. Therefore, we must find a choice of fundamental variables for this system which removes the non-homogeneous terms.

Fortunately, there exists a choice of fundamental variables for equations (57*a*)–(57*g*) which at the same time symmetrizes the principal matrices and eliminates the appearance of the undifferentiated function P, thus taking care of the two most important deficiencies of the system of evolution for the derivatives. Instead of using the seven first derivatives as fundamental variables, we can use the following:

$$\widehat{R}^{\phi} \equiv \frac{R_{\phi}}{r\sqrt{1-s^2}}, \qquad \widehat{R}^{s} \equiv \frac{R_s\sqrt{1-s^2}}{r}, \tag{58}$$

$$\widehat{P}^{\phi} \equiv \frac{P_{\phi}}{r\sqrt{1-s^2}}, \qquad \widehat{P}^{s} \equiv \frac{\partial_s\left(P\sqrt{1-s^2}\right)}{r} \tag{59}$$

$$\widehat{Q}^{\phi} \equiv \frac{Q_{\phi}}{r\sqrt{1-s^2}}, \qquad \widehat{Q}^{s} \equiv \frac{Q_s\sqrt{1-s^2}}{r}. \tag{60}$$

This re-scaling will necessarily contribute more undifferentiated terms to the system because of the dependence of the coefficients on r and s. In addition, the undifferentiated function P is absorbed into the fundamental variable $\widehat{P}^s$. In terms of the variables $w = (w_i) \equiv (\widehat{R}^s, \widehat{R}^{\phi}, R_r, \widehat{P}^s, \widehat{P}^{\phi}, \widehat{Q}^s, \widehat{Q}^{\phi})$, the system (57*a*)–(57*g*)has the form

$$\boldsymbol{B}^{a}\partial_a w = \boldsymbol{D}w, \tag{61}$$

where all the principal matrices $\boldsymbol{B}^{a}$ are symmetric and given explicitly by ${B^u}_{ij} = \mathrm{diagonal}(2, 2, 2, 0, 0, 0, 0)$, ${B^r}_{ij} = \mathrm{diagonal}(0, 0, -1, 1, 1, 1, 1)$ and ${B^s}_{ij} = {B^{\phi}}_{ij} = 0$ except

$${B^s}_{13} = {B^s}_{14} = {B^s}_{26} = -\frac{\sqrt{1-s^2}}{r} \tag{62}$$

$${B^{\phi}}_{15} = {B^s}_{23} = {B^s}_{27} = -\frac{1}{r\sqrt{1-s^2}}. \tag{63}$$

The matrix $\boldsymbol{D}$ has coefficients that depend on r and s but not on the unknown variables. The explicit expressions of D_{ij} are of no relevance for our purposes because they are sufficiently generic to force us to consider the most general case. Multiplying by w on the left, equation (61) implies

$$\partial_a(w\boldsymbol{B}^{a}w) = w\tilde{\boldsymbol{D}}w, \tag{64}$$

with $\tilde{\boldsymbol{D}} = 2\boldsymbol{D} + \partial_a\boldsymbol{B}^{a}$. Integrating (64) on the volume $\mathcal{V}^4$ enclosed by Σ_T, Σ_u and Σ_r we find

$$\int_{\Sigma_T} w(\boldsymbol{B}^{u} + \boldsymbol{B}^{r})w\,\mathrm{d}\Sigma_T - \int_{\Sigma_u} w\boldsymbol{B}^{u}w\,\mathrm{d}\Sigma_u - \int_{\Sigma_r} w\boldsymbol{B}^{r}w\,\mathrm{d}\Sigma_r = \int_{\mathcal{V}^4} w\tilde{\boldsymbol{D}}w\,\mathrm{d}\mathcal{V}^4. \tag{65}$$

We define

$$\|w\|_T^2 = \int_{\Sigma_T} w^2\,\mathrm{d}\Sigma_T \tag{66}$$

with $w^2 \equiv \sum_i w_i^2$. Then, exactly as in section 3.2, and because $\boldsymbol{B}^u + \boldsymbol{B}^r$ is the identity matrix, equation (65) implies

$$\|w\|_T \leqslant 2\left(\int_{\Sigma_u} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_u + \int_{\Sigma_r} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_r\right) + \int_{\mathcal{V}^4} w\tilde{\boldsymbol{D}}w\,\mathrm{d}\mathcal{V}^4. \tag{67}$$

The presence of the undifferentiated terms is a complication to the estimate of w in terms of the free data on both surfaces. This complication, however, can be resolved in a manner analogous as the case of the initial value problem dealt with in section 2. First note that

$$w\tilde{\boldsymbol{D}}w \leqslant cw^2 \tag{68}$$

where $c = \max(|\tilde{D}_{ij}|)$ in the volume $\mathcal{V}^4$. Thus $\int_{\mathcal{V}^4} w\tilde{\boldsymbol{D}}w\,\mathrm{d}\mathcal{V}^4 \leqslant c\int_0^T \mathrm{d}t \int_{\Sigma_t} w^2\,\mathrm{d}\Sigma_t$. With this, the inequality (67) implies

$$\|w\|_T^2 \leqslant 2\left(\int_{\Sigma_u} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_u + \int_{\Sigma_r} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_r\right) + c\int_0^T \|w\|_t^2\,\mathrm{d}t. \tag{69}$$

This holds for any fixed value of T, where Σ_u and Σ_r extend as far as their intersection with the surface at fixed value of $u + r = T$. If we denote $\Sigma_{u,t}$ and $\Sigma_{r,t}$ the regions of Σ_u and Σ_r extending only so far as their intersection with $u + r = t \leqslant T$, we can write

$$\begin{aligned}\|w\|_t^2 &\leqslant 2\left(\int_{\Sigma_{u,t}} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_{u,t} + \int_{\Sigma_{r,t}} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_{r,t}\right) + c\int_0^t \|w\|_{t'}^2\,\mathrm{d}t' \\ &\leqslant 2\left(\int_{\Sigma_u} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_u + \int_{\Sigma_r} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_r\right) + c\int_0^t \|w\|_{t'}^2\,\mathrm{d}t',\end{aligned} \tag{70}$$

which holds for any value of $t \leqslant T$. Using this inequality recursively into the right-hand side of (69) we have

$$\begin{aligned}\|w\|_T^2 \leqslant 2\left(1 + cT + \frac{c^2T^2}{2} + \cdots + \frac{c^mT^m}{m!}\right)&\left(\int_{\Sigma_u} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_u\right. \\ &\left.+ \int_{\Sigma_r} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_r\right) \\ &+ c^{m+1}\int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_m} \|w\|_{t_{m+1}}^2\,\mathrm{d}t_{m+1}\end{aligned} \tag{71}$$

for any given non-negative integer m. In the limit for $m \to \infty$ the sequence in the right-hand side converges if $cT < 1$, in which case we have

$$\|w\|_T^2 \leqslant 2e^{cT}\left(\int_{\Sigma_u} R_r^2 + \widehat{R}^{s\,2} + \widehat{R}^{\phi\,2}\,\mathrm{d}\Sigma_u + \int_{\Sigma_r} \widehat{P}^{s\,2} + \widehat{P}^{\phi\,2} + \widehat{Q}^{s\,2} + \widehat{Q}^{\phi\,2}\,\mathrm{d}\Sigma_r\right). \tag{72}$$

This is the final estimate for the derivatives in terms of the derivatives of the free data on both surfaces. Like in the case of the initial value problem dealt with in section 2, the estimate involves an exponential factor, essentially due to the presence of undifferentiated terms in the system of equations for w. As usual in such cases, even though the estimate is useful in order to prove that the solution depends continuously on the data for any value of T, it is impractical for large T for the purpose of estimating the error in a numerical solution. In particular, our proof only guarantees the estimate for $T < c^{-1}$. Perhaps with greater care, possibly by using Gronwall's inequality in integral form, the estimate could be extended to longer values of T.

Since the seven derivatives of interest can be found and estimated independently of P_u and Q_u, they can now be used as known sources for equations (57h)–(57i) written in terms of the new fundamental variables. The solutions P_u and Q_u can be obtained by quadratures as

$$P_u = \frac{r_0}{r} P_u|_{r_0} + \frac{\sqrt{1-s^2}}{2r} \partial_s \int_{r_0}^{r} R_r + \widehat{P}^s + \widehat{Q}^\phi \, \mathrm{d}r', \tag{73}$$

$$Q_u = \frac{r_0}{r} Q_u|_{r_0} + \frac{1}{2r\sqrt{1-s^2}} \partial_\phi \int_{r_0}^{r} R_r + \widehat{P}^s + \widehat{Q}^\phi \, \mathrm{d}r', \tag{74}$$

from the known functions and from free data given on $r = r_0$.

5. Concluding remarks

Summarizing, sections 3 and 4 develop the proofs of the following two theorems:

Theorem 1. *Consider equations (19a)–(19c), representing the first-order reduction of the wave equation in three spatial Cartesian coordinates* (x, y, z) *and one null coordinate* $u = t - z$. *Given data* $R|_{u=0} = f(x, y, z)$ *on the null surface* Σ_u *at* $u = 0$ *with* $0 \leqslant z \leqslant T$, $P|_{z=0} = g(u, x, y)$ *and* $Q|_{z=0} = h(u, x, y)$ *on the time-like surface* Σ_z *at* $z = 0$ *with* $0 \leqslant u \leqslant T$, *the unique solution* $v = (R, P, Q)$ *periodic in* (x, y) *satisfies the estimate*

$$\|v\|_T^2 \leqslant 2\left(\int_{\Sigma_u} R^2 \, \mathrm{d}\Sigma_u + \int_{\Sigma_z} (P^2 + Q^2) \, \mathrm{d}\Sigma_z\right)$$

with $\|v\|_T^2 \equiv \int_{\Sigma_T} (R^2 + P^2 + Q^2) \, \mathrm{d}\Sigma_T$, *where* Σ_T *is the space-like surface* $u + z = T$ *for* $0 \leqslant u \leqslant T$ *and* $0 \leqslant z \leqslant T$.

The derivatives of v *are similarly bounded by the derivatives of* f, g *and* h.

Theorem 2. *Consider equations (43a)–(43c), representing the first-order reduction of the wave equation in three spatial spherical coordinates* $(r, s = \cos\theta, \phi)$ *and one null coordinate* $u = t - r$. *Given data* $R|_{u=0} = f(r, s, \phi)$ *on the null surface* Σ_u *at* $u = 0$ *with* $r_0 \leqslant r \leqslant T + r_0$, $P|_{r=r_0} = g(u, s, \phi)$ *and* $Q|_{r=r_0} = h(u, s, \phi)$ *on the time-like surface* Σ_r *at* $r = r_0$ *with* $0 \leqslant u \leqslant T$, *the unique solution* $v = (R, P, Q)$ *satisfies the estimate*

$$\|v\|_T^2 \leqslant 2\left(\int_{\Sigma_u} R^2 \, \mathrm{d}\Sigma_u + \int_{\Sigma_z} (P^2 + Q^2) \, \mathrm{d}\Sigma_r\right)$$

with $\|v\|_T^2 \equiv \int_{\Sigma_T} (R^2 + P^2 + Q^2) \, \mathrm{d}\Sigma_T$, *where* Σ_T *is the space-like surface* $u + r = T + r_0$ *for* $0 \leqslant u \leqslant T$ *and* $r_0 \leqslant r \leqslant T + r_0$.

The derivatives of v *are similarly bounded by the derivatives of* f, g *and* h *for small values of* T.

These results are relevant to the stability of the solutions of the wave equation constructed from data given on two intersecting transverse surfaces, one of which is time-like and the other one is characteristic. The theorems guarantee that the solutions will be stable under small perturbations of the data on such two surfaces. Since the existence and uniqueness of solutions is already guaranteed by Duff's theorem [2], our result generalizes the standard notion of well posedness, available in the context of Cauchy problems, to the characteristic problem of the wave equation.

In the wider context of general interest in characteristic problems of any kind, three publications posterior to Duff's pioneering work [2] stand out for relevance and motivation.

In the first place, Müller zum Hagen and Seifert [6] recognized the value of energy estimates and correctly characterize the data surfaces and their role in the estimates for all types of problems, including problems with one or more characteristic surfaces. Perhaps due to the generic nature of their work, their estimates appear to have treated indifferently the normal data and the null data, making no distinction between free data and data that propagates within each characteristic surfaces. This problem is pointed out by Rendall [7], who, in reference to the Müller zum Hagen and Seifert work, says that 'they attempted, not entirely successfully, to give an existence and uniqueness proof by following step by step the treatment of the Cauchy problem'. Rendall proceeds to detail arguments for existence, uniqueness and stability for symmetric hyperbolic systems—like the wave equation—in the case of two characteristic transverse data surfaces. Rendall does conclude that estimates exist on the basis of the well posedness of the associated Cauchy problem, but unfortunately, offers no explicit estimates of the solution in terms of the set of free data, which he identifies. Balean [8–10] later calculates estimates of the energy kind for the wave equation with one characteristic and one time-like data surface where the null data are treated as a flow of information across a time-like boundary, and the distinction between the contributions of the free data and the transported data is not markedly emphasized. All three works deal exclusively with second-order equations, leaving Duff's systematic approach to first-order characteristic problems without a follow-up—in a strictly formal sense. In fact, we have been unable to identify any other published literature dealing with estimates for characteristic problems of any kind. Sometimes the second-order formulation of a problem and its associated first-order reduction are regarded as equally valid and interchangeable, but we find that new insights are often to be gained when a problem (even a well-understood one) is viewed from a different vantage point, a motivation that underlies our current work.

The method used here to address the question of stability is quite sensitive to the presence of undifferentiated terms and is likely to be sensitive to the presence of nonlinear terms as well. Still, its strength lies in its conceptual features, which depart quite significantly from the three predecessors already referred to. It is to be hoped that our conceptual framework will be useful as a general guideline for other characteristic problems. Work is in progress [11] to extend the method to generic linear characteristic problems for first-order systems of equations in order to obtain a criterion for 'manifest well posedness' that would play a role analogous to that of symmetric hyperbolicity of Cauchy problems. Our ultimate goal is to develop new insights into the nature of the characteristic problem of the Einstein equations in the form pioneered by Sachs [3]. In this respect, see [12–17] and also [4] for a review including the numerical implementation.

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